Gravothermal Catastrophe and Tsallis' Generalized Entropy of Self-Gravitating Systems II. Thermodynamic properties of stellar polytrope

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Abstract

In this paper, we continue to investigate the thermodynamic properties of stellar self-gravitating system arising from the Tsallis generalized entropy. In particular, physical interpretation of the thermodynamic instability, as has been revealed by previous paper(Taruya & Sakagami, Physica A 307 (2002) 185), is discussed in detail based on the framework of non-extensive thermostatistics. Examining the Clausius relation in a quasi-static experiment, we obtain the standard result of thermodynamic relation that the physical temperature of the equilibrium non-extensive system is identified with the inverse of the Lagrange multiplier, $T_{\rm phys} = 1/\beta$. Using this relation, the specific heat of total system is computed, and confirm the common feature of self-gravitating system that the presence of negative specific heat leads to the thermodynamic instability. In addition to the gravothermal instability discovered previously, the specific heat shows the curious divergent behavior at the polytrope index $n > 3$, suggesting another type of thermodynamic instability in the case of the system surrounded by the thermal bath. Evaluating the second variation of free energy, we check the condition for onset of this instability and find that the zero-eigenvalue problem of the second variation of free energy exactly recovers the marginal stability condition indicated from the specific heat. Thus, the stellar polytropic system is consistently characterized by the non-extensive thermostatistics as a plausible thermal equilibrium state. We also clarify the non-trivial scaling behavior appeared in specific heat and address the origin of non-extensive nature in stellar polytrope.

Key words: non-extensive entropy, self-gravitating system, gravothermal instability, negative specific heat, stellar polytrope
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1 Introduction

Due to its complexity and peculiarity, stellar self-gravitating system has long attracted much attention in the subject of astronomy and astrophysics, and even statistical physics. For an isolated stellar system, the dynamical equilibrium is rapidly attained after a few crossing time and the thermodynamic description provides useful information in characterizing the late-time behavior of this system. Even in this simplest situation, however, the equilibrium state of self-gravitating system shows various interesting phenomena, which may offer an opportunity to recast the framework of the thermodynamics and/or statistical mechanics.

In earlier paper, applying the Tsallis' generalized entropy[1], we have studied the thermodynamic instability of self-gravitating systems[2]. The self-gravitating stellar system confined in a spherical cavity of radius, r_e, exhibits an instability, so-called *gravothermal catastrophe*, which has been widely accepted as a fundamental physical process and plays an important role for the long-term evolution of globular clusters [3–5]. The presence of this instability has been long known since the pioneer work by Antonov[6] and Lynden-Bell & Wood[7]. Historically, the gravothermal catastrophe has been studied on the basis of the maximum entropy principle for the phase-space distribution function, with a particular attention to the Boltzmann-Gibbs entropy [8,9].

In contrast to previous work, we have applied the Tsallis-type generalized entropy to seek the equilibrium criteria for the first time (for comprehensive review of Tsallis formalism and its application to other field of physics, see [10,11]). Then, the distribution function of Vlassov-Poisson system can be reduced to a stellar polytropic system[12,13]. Evaluating the second variation of entropy around the equilibrium state and solving the zero-eigenvalue problem, the criterion for the onset of gravothermal instability is obtained. The main results of our previous analysis are summarized as follows:

(i) Local entropy extremum ceases to exist in cases with polytrope index $n > 5$ for sufficiently larger radius of the wall, $r_e > \lambda_{\rm crit}\, GM^2/(-E)$, and for highly density contrast, $\rho_c/\rho_e > D_{\rm crit}$, where M and E denote the total mass and energy of the system, ρ_c and ρ_e mean the density at center and edge, respectively.
(ii) The critical values $\lambda_{\rm crit}$ and $D_{\rm crit}$ depend on the polytrope index, both of which respectively approach 0.335 and 709 in the limit of $n \to \infty$, consistent with the well-known result adopting the Boltzmann-Gibbs entropy.
(iii) The stability/instability criterion obtained from the second variation of

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Tsallis entropy exactly matches with the result from standard turning-point analysis.

While the successful results suggest that non-extensive generalization of thermodynamics will offer various astrophysical applications involving long-range nature of self-gravitating systems, there still remain some important issues concerning the physical interpretation of thermodynamic instability.

Heuristically, the gravothermal instability is explained by the presence of negative specific heat as follows. In a fully relaxed gravitating system with sufficiently larger radius, negative specific heat arises at the inner part of the system and we have $C_{\rm V,inner} < 0$, while the specific heat at the outer part remains positive, $C_{\rm V,outer} > 0$, since one can safely neglect the effect of self-gravity. In this situation, if a tiny heat flow is momentarily supplied from inner to outer part, both the inner and the outer parts get hotter after the hydrostatic readjustment. Now imagine the case, $C_{\rm V,outer} > |C_{\rm V,inner}|$. The outer part has so much thermal inertia that it cannot heat up as fast as the inner part, and thereby the temperature difference between inner and outer parts increases. As a consequence, the heat flow never stops, leading to a catastrophe temperature growth.

While the above thought experiment is naive in a sense that we artificially divide the system into the inner and the outer part, the argument turns out to capture an essence of the thermodynamic instability in cases with the Boltzmann-Gibbs entropy. Evaluating the specific heat explicitly, Lynden-Bell and Wood[7] showed that the specific heat of the total system should be greater than zero at the onset of instability, although the central part of this system still has the negative specific heat. Therefore, one can naively expect that the self-gravitating system generally exhibits the thermodynamic instability associated with the negative specific heat and this could even hold in the system characterized by the non-extensive entropy.

To address this issue, however, we should remember the following two remarks that have been never clarified. First note that there exists a subtle point concerning the concept of temperature in the non-extensive thermodynamics. Framework of the non-extensive formalism is formally constructed keeping the standard result of thermodynamic relations [16–18], however, the physical temperature, $T_{\rm phys}$, might not be simply related to the usual one, i.e, the inverse of Lagrange multiplier, as has been criticized recently[14,15]. This point is in particular important in evaluating the specific heat.

Second, as has been mentioned by the pioneer work of Lynden-Bell & Wood[7], self-gravitating system shows various types of thermodynamic instability. While our early study deals with the stellar system confined within an adiabatic wall,

one may replace the adiabatic wall with the thermally conducting wall surrounded by a heat bath. In this situation, assuming the Boltzmann-Gibbs entropy, Lynden-Bell & Wood showed that no equilibrium state exists for sufficiently low temperature and high-density contrast. Note that even in this case, the presence of negative specific heat plays an essential role for the appearance of instability.

Keeping the above remarks in mind, in this paper, we focus on the thermodynamic property of self-gravitating systems characterized by Tsallis' generalized entropy. For this purpose, we first investigate the thermodynamic temperature of the self-gravitating system from the Clausius relation. To clarify the physical interpretation of thermodynamic instability, the specific heat is computed and a role of negative specific is discussed in detail. Then we turn to focus on the thermodynamic instability in a system surrounded by the heat bath. The stability/instability criterion is derived from the second variation of free energy and a geometrical construction of marginal stability condition is discussed.

While the problem considered here includes some general issues that are commonly faced with the application of the non-extensive thermostatistics, we will tackle this problem based on the *old* Tsallis formalism using the standard statistical average, which is currently un-common (e.g., [14,15][18]). The reason why we do not adopt the *standard* Tsallis formalism using the normalized q-expectation values is twofold. As has been mentioned in previous paper, a naive application of the new formalism apparently shows a problematic difficulty in our case of the maximum entropy principle (Sec.2), while no such difficulties arise when we apply the earlier formalism (see ref.[2] in details). Another reason is that while the new formalism has been deliberately constructed so as to eliminate the undesirable divergences in some physical systems [18] especially with fractal nature, no serious divergences have appeared in our case. Precisely speaking, the physical quantities, e.g. mass and energy, may have divergence for some equilibrium configuration of the self-gravitating system. In order to remedy this divergence, we confine the system within a spherical wall, which is a standard prescription in studies for the self-gravitating system [6,7]. Since even the old Tsallis formalism preserves a consistent framework that recovers the usual thermodynamic structure, from a more general view of the non-extensive thermostatistics, we expect that the present analysis still provides a valuable insight to the thermodynamic stability of stellar self-gravitating system. Of course, the analysis using normalized q-value must play an important role in the Tsallis' non-extensive framework and we plan to extend our analysis to the one with the new formulation near future. In this sense, present work can be regarded as a preliminary analysis toward the next step. This point will be discussed in the last part of this paper, together with some implications.

This paper is organized as follows. in section 2, we recast the problem that finds

the most probable state of equilibrium stellar distribution adopting the Tsallis entropy. The main part of this paper is section 3, in which the thermodynamic properties of stellar polytrope are investigated in detail. After identification of the thermodynamic temperature, the explicit expression for specific heat is presented and the marginally stability condition for the thermodynamic instability is investigated in both the adiabatic and the isothermal cases. In section 4, thermodynamic instability in a system surrounded by a thermal bath is re-considered by means of the free energy and the marginal stability condition is re-derived from the second variation of free energy. Furthermore, following the preceding results, the origin of the non-extensive nature in stellar polytropic system is discussed in section 5. Finally, section 6 is devoted to the summary and discussions.

2 Stellar polytrope as an extremum state of Tsallis entropy

In this section, we recast the problem finding the most probable state of equilibrium stellar system, based on the maximum entropy principle. In our previous study, the entropy for the phase-space distribution function has been introduced without recourse to the correct dimensions. Although this does not alter the stability/instability criterion for the stellar equilibrium state, for the sake of the completeness and the later analysis, we repeat the same calculation as shown in ref.[2], taking fully account of the correct dimensions.

Suppose a system containing N particles which are confined within a hard sphere of radius r_e. For simplicity, each particle is assumed to have the same mass m_0 and interacts via Newton gravity. The problem considered here is to find an equilibrium state in an adiabatic treatment. That is, we investigate the equilibrium particle distribution in which the particles elastically bounce from the wall, keeping the energy E and the total mass $M(=Nm_0)$ constant.

For present purpose, it is better to employ the mean-field treatment that the correlation between particles is smeared out and the system can be fully characterized by the one-particle distribution function, $f(\boldsymbol{x},\boldsymbol{v})$, defined in six-dimensional phase-space $(\boldsymbol{x},\boldsymbol{v})$ [2,3][6–9]. Let us denote the phase-space element as $h^3(=l_0^3v_0^3)$ with unit length l_0 and unit velocity v_0. Since the distribution function $f(\boldsymbol{x},\boldsymbol{v})$ counts the number of particles in a unit cell of phase-space, using the standard definition of the statistical average, the energy and the total mass are respectively expressed as follows:

$$E = K + U \;\equiv\; m_0 \int \left\{\frac{1}{2}\,v^2 + \frac{1}{2}\,\Phi(\boldsymbol{x})\right\} f(\boldsymbol{x},\boldsymbol{v})\; d^6\boldsymbol{\tau}, \tag{1}$$

$$M = m_0 N \equiv m_0 \int f(\boldsymbol{x}, \boldsymbol{v}) \; d^6\boldsymbol{\tau} \;, \tag{2}$$

with the quantity Φ being the gravitational potential:

$$\Phi(\boldsymbol{x}) = -G \, m_0 \int \frac{f(\boldsymbol{x}', \boldsymbol{v}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \; d^6\boldsymbol{\tau}'. \tag{3}$$

In the above expressions, the dimensionless integral measure $d^6\boldsymbol{\tau}$ is introduced:

$$d^6\boldsymbol{\tau} \equiv \frac{d^3\boldsymbol{x} \, d^3\boldsymbol{v}}{h^3} \quad ; \; h = l_0 \, v_0. \tag{4}$$

Owing to the maximum entropy principle, we explore the most probable state maximizing the entropy. The entropy quoted here is a quantity defined in the phase-space and it counts the number of possible particle state. We are specifically concerned with the equilibrium state for the Tsallis entropy [1]:

$$S_q = -\frac{N}{q-1} \int \left[\left(\frac{f}{N} \right)^q - \left(\frac{f}{N} \right) \right] \; d^6\boldsymbol{\tau}. \tag{5}$$

Maximizing the entropy S_q under the constraints reduces to the following mathematical problem using Lagrange multipliers α and β:

$$\delta S_q - \alpha \, \delta M - \beta \, \delta E = 0, \tag{6}$$

which leads to [2,12,13]:

$$f(\boldsymbol{x}, \boldsymbol{v}) \;=\; A \, \left[\Phi_0 - \Phi(\boldsymbol{x}) - \frac{1}{2} v^2 \right]^{1/(q-1)} , \tag{7}$$

where the constants A and Φ_0 are respectively given by

$$A = N \left\{ \left(\frac{q-1}{q} \right) m_0 \beta \right\}^{1/(q-1)} , \qquad \Phi_0 = \frac{1 - (q-1) m_0 \, \alpha}{(q-1) m_0 \, \beta}. \tag{8}$$

The one-particle distribution function (7) is often called *stellar polytrope*, which satisfies the polytropic equation of state [3][12]. The density profile $\rho(r)$ and the isotropic pressure $P(r)$ at the radius $r = |\boldsymbol{x}|$ are respectively given by

$$\rho(r) \equiv m_0 \int f(\boldsymbol{x}, \boldsymbol{v}) \; \frac{d^3\boldsymbol{v}}{h^3}$$

$$= 4\sqrt{2}\pi \; B\left(\frac{3}{2}, \frac{q}{q-1}\right) \; \frac{m_0 \, A}{h^3} \; \{\Phi_0 - \Phi(r)\}^{1/(q-1)+3/2}, \tag{9}$$

and

$$\begin{aligned} P(r) &\equiv m_0 \int \frac{1}{3} v^2 \, f(\boldsymbol{x}, \boldsymbol{v}) \; \frac{d^3\boldsymbol{v}}{h^3} \\ &= \left(\frac{1}{q-1} + \frac{5}{2}\right)^{-1} \; \rho(r) \; \{\Phi_0 - \Phi(r)\}, \end{aligned} \tag{10}$$

with $B(a, b)$ being the β function. Thus, these two equations lead to the relation

$$P(r) = K_n \, \rho^{1+1/n}(r), \tag{11}$$

with the polytrope index given by

$$n = \frac{1}{q-1} + \frac{3}{2}. \tag{12}$$

In equation (11), the dimensional constant K_n is introduced:

$$K_n \equiv \frac{1}{n+1} \left\{4\sqrt{2}\pi B\left(\frac{3}{2}, n - \frac{1}{2}\right) \; \frac{m_0 \, A}{h^3}\right\}^{-1/n}. \tag{13}$$

Note that the above quantity is equivalent to the variable $(n - 3/2)T/(n + 1)$ defined in ref.[2].

Once provided the distribution function, the equilibrium configuration can be completely specified by solving the Poisson equation. Hereafter, we specifically restrict our attention to the spherically symmetric configuration for $q > 1$(or $n > 3/2$), in which the dynamically stable state is safely attainable and the thermodynamic arguments turn out to capture the physical relevance [3].

From the gravitational potential (3), it reads

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi(r)}{dr}\right) = 4\pi G \rho(r). \tag{14}$$

Combining (14) with (9), we obtain the ordinary differential equation for Φ. Equivalently, a set of equations which represent the hydrostatic equilibrium are derived using (9), (10) and (14):

$$
\begin{aligned}
\frac{dP(r)}{dr} &= -\frac{Gm(r)}{r^2}\,\rho(r), \qquad (15)\\
\frac{dm(r)}{dr} &= 4\pi\rho(r)\,r^2. \qquad (16)
\end{aligned}
$$

The quantity $m(r)$ denotes the mass evaluated at the radius r inside the wall. We then introduce the dimensionless quantities:

$$
\rho = \rho_c\,[\theta(\xi)]^n\,, \quad r = \left\{\frac{(n+1)P_c}{4\pi G\rho_c^2}\right\}^{1/2}\xi, \qquad (17)
$$

which yields the following ordinary differential equation:

$$
\theta'' + \frac{2}{\xi}\theta' + \theta^n = 0, \qquad (18)
$$

where prime denotes the derivative with respect to ξ. The quantities ρ_c and P_c in (17) are the density and the pressure at $r = 0$, respectively. To obtain the physically relevant solution of (18), we put the following boundary condition:

$$
\theta(0) = 1, \quad \theta'(0) = 0. \qquad (19)
$$

A family of solutions satisfying (19) is referred to as the *Emden solution*, which is well-known in the subject of stellar structure (e.g., see Chap.IV of ref.[19]).

Figure 1 shows the numerical solution of equation (18) for various polytrope indices, where the density profile, $\rho(r)/\rho_c$ is plotted as a function of dimensionless radius, ξ. Clearly, profiles with index $n < 5$ rapidly fall off and they abruptly terminate at finite radius(*left-panel*), while the $n \geq 5$ cases infinitely continue to extend over the outer radius(*right-panel*). As already mentioned in previous paper, characteristic feature seen in figure 1 plays an essential role for the thermodynamic instability associated with negative specific heat.

For later analysis, it is convenient to introduce the following set of variables, referred to as homology invariants [19,20]:

$$
\begin{aligned}
u &\equiv \frac{d\ln m(r)}{d\ln r} = \frac{4\pi r^3\rho(r)}{m(r)} = -\frac{\xi\theta^n}{\theta'}, \qquad (20)\\
v &\equiv -\frac{d\ln P(r)}{d\ln r} = \frac{\rho(r)}{P(r)}\;\frac{Gm(r)}{r} = -(n+1)\frac{\xi\theta'}{\theta}, \qquad (21)
\end{aligned}
$$

which reduce the degree of equation (18) from two to one. The derivative of

these variables with respect to ξ becomes

$$\frac{du}{d\xi} = \left(3 - u - \frac{n}{n+1}\,v\right)\frac{u}{\xi}, \qquad \frac{dv}{d\xi} = \left(-1 + u + \frac{1}{n+1}\,v\right)\frac{v}{\xi}. \tag{22}$$

Equations (18) can thus be re-written with

$$\frac{u}{v}\frac{dv}{du} = \frac{(n+1)(u-1)+v}{(n+1)(3-u)-nv}. \tag{23}$$

The corresponding boundary condition to (19) becomes $(u,v)=(3,0)$. Using these variables, the basic thermodynamic quantities such as the energy and the entropy are evaluated and the results are summed up in Appendix A, which are subsequently used in section 3.

3 Thermodynamic properties of stellar polytrope

In this section, we address our main issue, i.e, the physical interpretation of gravothermal instability in stellar polytropes, based on the framework of non-extensive thermodynamics. In section 3.1, we first discuss the thermodynamic temperature of stellar polytrope calculating both the heat and the entropy changes in a quasi-static treatment. Then we evaluate the specific heat in section 3.2. The connection between the absence of extremum entropy state and the presence of negative specific heat is discussed in detail. Further, we argue that there appears another type of thermodynamic instability, which is subsequently analyzed by means of the free energy.

3.1 Thermodynamic temperature from the Clausius relation

As has been mentioned in section 1, the concept of temperature is non-trivial in non-extensive thermostatistics. This is because the standard framework of thermodynamics crucially depends on the assumption of extensivity of entropy. According to the recent claim, the definition of physical temperature $T_{\rm phys}$ should be altered depending on the choice of energy constraint and is related to the inverse of the Lagrange multiplier, $1/\beta$, with *some correction factors* [14,15]. Note, however, that this discussion heavily relies on the extensivity of the energy as well as the thermodynamic zeroth law. In our present case, the maximum entropy principle was applied subject to the constraints E and M, adopting the standard definition of mean values (see eqs.(1)(2)). As a consequence, the resultant energy E becomes non-extensive and we cannot apply the above definition.

To address the physical temperature in the present case, we therefore consider the relation between the heat transfer and entropy change and seek the most plausible candidate for thermodynamic temperature. That is, we analyze the variation of equilibrium configuration under fixing the total mass. Specifically, we deal with the quasi-static variation along an equilibrium sequence.

Let us first write down the heat change. The thermodynamic first law states that

$$d'Q \;=\; dE \,+\, P_e\, dV, \tag{24}$$

where the operation d' stands for incomplete differentiation. The subscript $_e$ denotes a quantity evaluated at the edge. In the spherically symmetric configuration, the second term in right-hand side of (24) becomes $4\pi r_e^2 P_e\, dr_e$. As for the first term, the energy of the stellar polytropic system within the radius r_e, is computed in Appendix A.1. Introducing the dimensionless parameter λ, it is expressed in terms of the homology invariants as follows:

$$\lambda \equiv -\frac{r_e E}{GM^2} \;=\; -\frac{1}{n-5}\,\left[\frac{3}{2}\left\{1-(n+1)\frac{1}{v_e}\right\}+(n-2)\frac{u_e}{v_e}\right], \tag{25}$$

where the quantity with subscript $_e$ represents the one evaluated at the boundary $r = r_e$. Using (25), the heat change $d'Q$ is rewritten as follows:

$$\begin{aligned} d'Q \;&=\; d\left(-\lambda\,\frac{GM^2}{r_e}\right) \,+\, 4\pi r_e^2\, P_e\, dr_e, \\ &=\; \frac{GM^2}{r_e}\,\left\{\left(\lambda+\frac{u_e}{v_e}\right)\,\frac{dr_e}{r_e}\,-\,\xi_e\,\frac{d\lambda}{d\xi_e}\,\frac{d\xi_e}{\xi_e}\right\}, \end{aligned} \tag{26}$$

where the relation $4\pi r_e^4 P_e/(GM^2) = u_e/v_e$ is used in the last line (see definitions (20)(21)). In the above expression, derivative of λ with respect to ξ_e can be computed with a help of relation (22) (see eq.(33) of ref.[2]):

$$\xi_e\frac{d\lambda}{d\xi_e} \;=\; \frac{n-2}{n-5}\;\frac{g(u_e,\,v_e)}{2v_e}, \tag{27}$$

where

$$g(u,v) = 4u^2 + 2uv - \left\{8 + 3\left(\frac{n+1}{n-2}\right)\right\}u - \frac{3}{n-2}\,v + 3\left(\frac{n+2}{n-2}\right). \tag{28}$$

Next focus on the change of the entropy. From (71) in Appendix A.2, the

entropy of the extremum state is given by

$$S_q = \left(n - \frac{3}{2}\right)\left[\frac{1}{n-5}\,\frac{\beta G M^2}{r_e}\,\left\{2\frac{u_e}{v_e} - (n+1)\frac{1}{v_e} + 1\right\} + N\right]. \tag{29}$$

Hence, the variation of entropy dS_q under fixing the total mass can be decomposed into the variation of homology invariants (u_e, v_e), radius r_e and Lagrange multiplier β as follows:

$$\begin{aligned} dS_q = \frac{n-3/2}{n-5}\,\frac{\beta G M^2}{r_e}\,&\left[\left(\frac{d\beta}{\beta} - \frac{dr_e}{r_e}\right)\left\{2\frac{u_e}{v_e} - (n+1)\frac{1}{v_e} + 1\right\}\right. \\ &\left. + \left\{2\frac{u_e}{v_e}\left(\frac{du_e}{u_e} - \frac{dv_e}{v_e}\right) - \frac{n+1}{v_e}\frac{dv_e}{v_e}\right\}\right]. \end{aligned} \tag{30}$$

Among these variations, variation of homology invariants is simply rewritten with $d\xi_e$, through the relation (22). On the other hand, from the mass conservation, the variation of Lagrange multiplier, $d\beta$ is related to both the variations of homology invariants and dr_e as follows. Using the condition of hydrostatic equilibrium at the edge r_e, one can obtain the following relation (see derivation in Appendix A.3):

$$\eta \equiv \left\{\frac{(GM)^n (m_0\beta)^{n-3/2}}{r_e^{n-3} h^3}\right\}^{1/(n-1)} = \alpha_n\ (u_e\, v_e^n)^{1/(n-1)}, \tag{31}$$

where the constant α_n is given by

$$\alpha_n = \left\{\frac{(n-1/2)^{n-3/2}}{16\sqrt{2}\pi^2\,(n+1)^n\,B(3/2, n-1/2)}\right\}^{1/(n-1)}, \tag{32}$$

which asymptotically approaches unity, in the limit $n \to +\infty$. Keeping the total mass M constant, variation of (31) yields

$$\frac{n-3/2}{n-1}\,\frac{d\beta}{\beta} - \frac{n-3}{n-1}\,\frac{dr_e}{r_e} = \frac{1}{n-1}\left(\frac{du_e}{u_e} + n\,\frac{dv_e}{v_e}\right). \tag{33}$$

We then rewrite it with

$$\frac{d\beta}{\beta} - \frac{dr_e}{r_e} = \frac{1}{n-3/2}\,\left(-\frac{3}{2}\frac{dr_e}{r_e} + \frac{du_e}{u_e} + n\,\frac{dv_e}{v_e}\right). \tag{34}$$

Substituting the relation (34) into equation (30), the dependence of $d\beta/\beta$ can be eliminated. Thus, using the relation (22), the final form of the entropy

change is expressed in terms of the variations $d\xi_e$ and dr_e. After some manipulation, we obtain

$$dS_q = \frac{\beta G M^2}{r_e}\left[-\frac{3/2}{n-5}\left(2\,\frac{u_e}{v_e}-\frac{n+1}{v_e}+1\right)\frac{dr_e}{r_e}-\frac{n-2}{n-5}\,\frac{1}{2v_e}\right.$$
$$\left.\times\left\{4u_e^2+2u_e v_e-\left(8+3\frac{n+1}{n-2}\right)u_e-\frac{3}{n-2}\,v_e+3\left(\frac{n+1}{n-2}\right)\right\}\frac{d\xi_e}{\xi_e}\right]. \quad (35)$$

Now, from the knowledge of the expressions λ and $\xi_e(d\lambda/d\xi_e)$, one can easily show that the above equation is just identical to

$$dS_q = \frac{\beta G M^2}{r_e}\,\left\{\left(\lambda+\frac{u_e}{v_e}\right)\frac{dr_e}{r_e}-\xi_e\,\frac{d\lambda}{d\xi_e}\,\frac{d\xi_e}{\xi_e}\right\}. \quad (36)$$

Therefore, comparison between (36) and (26) immediately leads to the following relation:

$$dS_q \;=\; \beta\, d'Q = \beta\,(dE + P_e dV), \quad (37)$$

which exactly coincides with the standard result of *Clausius relation* in a quasi-static process.

The relation (37) strongly suggests that the thermodynamic temperature $T_{\rm phys}$ is identified with the inverse of Lagrange multiplier, $T_{\rm phys} = 1/\beta$. At first glance, the result seems somewhat trivial, since one can easily expect this relation from the standard thermodynamic relation, $\partial S_q/\partial E = \beta$, which generally holds even in the the non-extensive Tsallis formalism [16,17]. As advocated by many author, however, the relation $\partial S_q/\partial E = \beta$ does not simply imply the thermodynamic temperature $T_{\rm phys} = 1/\beta$ and it might even contradict with the thermodynamic temperature defined through the thermodynamic zeroth low [15].

On the other hand, in our case of the self-gravitating system, the thermodynamic temperature $T_{\rm phys} = 1/\beta$ is mathematically verified by the integrable condition of the thermodynamic entropy through the Clausius relation. Further, it is remarkably found that the relation $T_{\rm phys} = 1/\beta$ holds even in the absence of gravity (the limit $G \to 0$) and can be proven through an alternative route. In Appendix B, as a pedagogical example, we demonstrate that the relation $T_{\rm phys} = 1/\beta$ is indeed obtained in the classical gas model using the Carnot cycle.

3.2 Negative specific heat and thermodynamic instability

Once obtained the thermodynamic temperature, $T_{\rm phys} = 1/\beta$, we are in a position to investigate the thermodynamic instability from the straightforward calculation of the specific heat. Let us first discuss the qualitative behavior of the specific heat. By definition, the specific heat at constant volume is given by

$$C_{\rm V} \equiv \left(\frac{dE}{dT_{\rm phys}}\right)_e = -\beta^2 \left(\frac{dE}{d\beta}\right)_e = -\beta^2 \frac{\left(\dfrac{dE}{d\xi}\right)_e}{\left(\dfrac{d\beta}{d\xi}\right)_e}. \tag{38}$$

Recall that the dimensionless parameters λ and η are respectively proportional to $-E$ and $\beta^{(n-1)/(n-3/2)}$ (see eqs.(25)(31)). This implies that for a system of constant mass inside a fixed wall, the qualitative behavior of (38) can be deduced from the relation between η and λ.

Figure 2 depicts the trajectories of the Emden solutions in the (η,λ)-plane with various polytrope indices. Each point along the trajectory represents an Emden solution for different value of the radius r_e. From the boundary condition, all the trajectories start from $(\eta,\lambda) = (0,-\infty)$, corresponding to the origin $r_e = 0$. As gradually increasing the radius, the trajectories first move to upper-right direction monotonically, as marked by the arrow. At this stage, the kinematic energy dominates the potential energy and the system lies in a kinematically thermal state ($\lambda < 0$), indicating the positive specific heat. For larger radius, while the curves with index $n \leq 3$ abruptly terminate, the trajectories with $n > 3$ suddenly change their direction from upper-right to upper-left. Moreover, in the case of $n > 5$, the trajectory progressively changes its direction and it finally spirals around a fixed point.

From these observations, one can roughly infer the existence of the two types of the thermodynamic instability as follows. At first inflection point for $n > 3$, the specific heat diverges and the signature of $C_{\rm V}$ becomes indefinite. Beyond this point, the specific heat changes from positive to negative. This means that the potential energy conversely dominates the kinetic energy, indicating the system being *gravothermal*. In this case, equilibrium state ceases to exist for a system in contact with a heat bath, but does still exist for a system surrounded by an adiabatic wall. However, for the polytrope index $n > 5$, the specific heat of the system turns to increase beyond this inflection point and it next reaches at the point $d\lambda/d\eta = 0$, i.e, $C_{\rm V} = 0$. This means that while the inner part of the system still keeps the specific heat negative, the fraction of the outer normal part grows up as increasing r_e and it eventually balances with inner gravothermal part. Thus, beyond this critical point, no thermal

balance is attainable and the system becomes gravothermally unstable. This is true even in the system surrounded by an adiabatic wall.

Now, let us write down the explicit expression for the specific heat $C_{\rm V}$. In equation (38), the variation of β and E with ξ_e can be respectively rewritten with

$$\left(\frac{dE}{d\xi}\right)_e = -\frac{GM^2}{r_e}\,\frac{d\lambda}{d\xi_e}, \tag{39}$$

and

$$\left(\frac{d\beta}{d\xi}\right)_e = \frac{n-1}{n-3/2}\,\frac{\beta}{\eta}\,\frac{d\eta}{d\xi_e}. \tag{40}$$

Here, the variable $d\lambda/d\xi_e$ has been already given in (27). As for the derivative of η with respect to ξ_e, we obtain

$$\xi_e\,\frac{d\eta}{\xi_e} = \left(u_e - \frac{n-3}{n-1}\right)\eta. \tag{41}$$

Then the quantity $C_{\rm V}$ becomes

$$C_{\rm V} = \frac{(n-3/2)(n-2)}{(n-1)(n-5)}\,\frac{\beta GM^2}{r_e}\,\frac{g(u_e,v_e)}{2v_e\left(u_e - \dfrac{n-3}{n-1}\right)},$$

with the function $g(u_e, v_e)$ given by (28). Notice that the above expression is still redundant, since there remains the explicit dependence of the variable β. Eliminating the variable β by using the relation (31), one finally obtains

$$\frac{C_{\rm V}}{N} = \tilde{\alpha}_n\left(\frac{h^2}{GMr_e}\right)^{(3/2)/(n-3/2)}\,\frac{g(u_e,v_e)}{2\left(u_e - \dfrac{n-3}{n-1}\right)}\,\left(u_e\,v_e^{3/2}\right)^{1/(n-3/2)}, \tag{42}$$

where we introduced the new dimensionless constant $\tilde{\alpha}_n$:

$$\tilde{\alpha}_n \equiv \frac{(n-3/2)(n-2)}{(n-1)(n-5)}\,\alpha_n^{1/(n-3/2)}. \tag{43}$$

Note that in the limit $n \to +\infty$, equation (42) consistently recovers the well-known result of isothermal sphere (e.g, eq.(39) of ref.[22]):

$$\frac{C_{\rm V}}{N} \;\xrightarrow{n\to+\infty}\; \frac{4u_e^2 + 2u_e v_e - 11u_e + 3}{2(u_e-1)}. \tag{44}$$

Comparing (42) with the isothermal limit, the resultant expression contains a residual dimensional parameter h, as well as the quantities M and r_e. While the residual dependence can be regarded as a natural consequence of the non-extensive generalization of the entropy, it would be helpful to understand the origin of this scaling in more simplified manner. This will be discussed in section 5.

Apart from the residual factor, the expression of specific heat (42) clearly reveals the two types of thermodynamic instability seen in Figure 2. The inflection point with the infinite specific heat, $C_{\rm V} \to \pm\infty$ leads to the condition

$$u_e - \frac{n-3}{n-1} = 0, \tag{45}$$

which immediately yields the conclusion that this is only possible for the polytrope index $n > 3$, consistent with Figure 2. On the other hand, critical point with the vanishing specific heat, $C_{\rm V} = 0$ corresponds to the following condition:

$$g(u_e,\, v_e) = 0. \tag{46}$$

This is exactly the same condition as obtained from the second variation of entropy (see eq.(33) or (53) in ref.[2]). According to the previous analysis, the condition (46) represents the marginal stability at which the extremum state of the entropy S_q is neither maximum nor minimum. This situation turns out to appear when the polytrope index $n > 5$.

Therefore, we reach a fully satisfactory conclusion that the thermodynamic instability found from the second variation of entropy is intimately related to the presence of negative specific heat and the stability/instability criterion can be exactly recovered from the critical point of the thermal balance, $C_{\rm V} = 0$, which is also consistent with the analysis in the Boltzmann-Gibbs limit, $n \to \infty$ [7]. The successful result can be regarded as an outcome of the correct definition of $T_{\rm phys}$. As for the transition point with $C_{\rm V} \to \pm\infty$, it clearly indicates the thermodynamic instability of a system in contact with a thermal bath. In next section, by means of the free energy, we confirm that the condition (45) indeed represents the marginal stability of the system surrounded by a thermal wall and beyond this point the system will be unstable.

In Figure 3, by varying the radius r_e, the normalized specific heat per particle $C_{\rm V}^*/N$ is plotted as a function of density contrast, ρ_c/ρ_e around the critical polytrope indices $n = 3$(*upper-panels*) and $n = 5$(*middle-panels*). Here, the normalized specific heat $C_{\rm V}^*$ is defined by the specific heat $C_{\rm V}$ divided by the redundant factor $(h^2/GMr_e)^{(3/2)/(n-3/2)}$. Obviously, the transition point $C_{\rm V} \to \pm\infty$ appears when $n > 3$(*crosses*), while the existence of critical point

$C_{\rm V} = 0$ is allowed for higher density contrast of $n > 5$ cases(*arrows*). The critical values $D_{\rm crit} \equiv (\rho_c/\rho_e)_{\rm crit}$ indicated by arrows exactly coincide with those obtained from the previous analysis (see Table 1 of ref.[2]). Lower-panels of Figure 3 show the specific heat with large polytrope indices $n = 10$ and 30, together with the Boltzmann-Gibbs limit ($n \to +\infty$, labeled by *iso*). As increasing the polytrope index n, the critical/transition points tend to shift to the lower density contrast, while the successive divergent and zero-crossing points appear at the higher density contrast, corresponding to the behavior seen in Figure 2.

4 Thermodynamic instability from the second variation of free energy

Previous section reveals that there exists another type of thermodynamic instability in which the marginal stability is deduced from the condition (45). In this section, to check the consistency of the non-extensive thermostatistics, we reconsider this issue by means of the Helmholtz free energy:

$$F_q = E - T_{\rm phys}\, S_q. \tag{47}$$

Adopting the relation $T_{\rm phys} = 1/\beta$, we re-derive the marginal stability condition (45) from the second variation of F_q.

Consider a system surrounded by the thermally conducting wall in contact with a heat bath. Usually, the stable equilibrium state should keep the free energy F_q minimum. Thus the presence of thermodynamic instability implies the absence of minimum free energy, which can be deduced from the signature of the second variation $\delta^2 F_q$ around the extremum state of free energy. Since the non-extensive formalism still verifies the Legendre transform structure leading to the standard result of thermodynamic relation[16,17], the extremum state of the free energy exactly coincides with that of the entropy. One thus skips to find the extremum state of F_q and proceeds to evaluate the second order variation.

In contrast to the adiabatic treatment, we here deal with the density perturbation $\rho \to \rho + \delta\rho$, surrounded by a thermal wall. To be specific, we evaluate the second variation under keeping the radius r_e, the total mass M and the temperature $T_{\rm phys}$ constant. Then the variation of energy up to the second order leads to

$$\delta E = \delta\left[\int \left\{\frac{3}{2}\,P(x) + \frac{1}{2}\rho(x)\Phi(x)\right\} d^3\boldsymbol{x}\right],$$

$$= \int \left\{ \frac{3}{2}\,\delta P + \frac{1}{2}(\delta\rho\,\Phi + \rho\,\delta\Phi) + \frac{1}{2}\,\delta\rho\,\delta\Phi \right\} d^3\boldsymbol{x}. \tag{48}$$

Similarly, using the expression (70) in Appendix A.2, the variation of Tsallis entropy becomes

$$\begin{aligned} \delta S_q &= \delta \left[\left(n - \frac{3}{2} \right) \left\{ N - \beta \int \; P(x)\, d^3\boldsymbol{x} \right\} \right], \\ &= - \left(n - \frac{3}{2} \right) \beta \int \delta\, P(x) d^3\boldsymbol{x}. \end{aligned} \tag{49}$$

The above expressions include the variation of pressure δP, which can be expanded with a help of the polytropic equation of state (11):

$$\delta P = \left(1 + \frac{1}{n} \right) \frac{P}{\rho}\,\delta\rho + \frac{1}{2} \left(1 + \frac{1}{n} \right) \frac{1}{n}\,\frac{P}{\rho^2}\,(\delta\rho)^2\,. \tag{50}$$

Combining the above result with equations (48) and (49) and collecting the second order terms only, the second variation of free energy becomes

$$\delta^2 F_q = \delta^2 E - T_{\rm phys}\,\delta^2 S_q = \frac{1}{2} \int \left\{ \frac{n+1}{n}\,\frac{P}{\rho^2}\,(\delta\rho)^2 \; + \; \delta\rho\,\delta\Phi \right\} d^3\boldsymbol{x}, \tag{51}$$

where the relation $T_{\rm phys} = 1/\beta$ is used in the last line. Now, restricting our attention to the spherical symmetric perturbation, we introduce the following perturbed quantity (see refs. [2][8]):

$$\delta\rho(r) = \frac{1}{4\pi r^2}\,\frac{dQ(r)}{dr}. \tag{52}$$

Then the mass conservation $\delta M = 0$ implies the boundary condition $Q(0) = Q(r_e) = 0$. Substituting (52) into (51) and repeating the integration by part, one finally reaches the following quadratic form:

$$\delta^2 F_q \;=\; -\frac{1}{2} \int_0^{r_e} dr\; Q(r) \left[\frac{n+1}{n}\,\frac{d}{dr} \left\{ \frac{1}{4\pi\,r^2\,\rho} \left(\frac{P}{\rho} \right) \frac{d}{dr} \right\} + \frac{G}{r^2} \right] Q(r). \tag{53}$$

Thus, the problem just reduces to the eigenvalue problem and the stability of the system can be deduced from the signature of the eigenvalue. More

specifically, the onset of instability corresponds to the marginally stability condition, $\delta^2 F_q = 0$, and it is sufficient to analyze the zero-eigenvalue equation:

$$\hat{L}\, Q(r) \equiv \left[\frac{d}{dr}\left\{\frac{1}{4\pi\, r^2\, \rho}\,\left(\frac{P}{\rho}\right)\frac{d}{dr}\right\} + \frac{n}{n+1}\,\frac{G}{r^2}\right]\, Q(r) \;=\; 0, \tag{54}$$

with the boundary condition, $Q(0) = Q(r_e) = 0$. Equation (54) has quite similar form to the zero-eigenvalue equation found in the adiabatic treatment (see eq.(46) of ref.[2]). Except for the non-local term, one can utilize the previous knowledge to solve the equation (54):

$$\hat{L}\,(4\pi r^3\,\rho) = \frac{n-3}{n+1}\,\frac{G\,m(r)}{r^2}, \qquad \hat{L}\,m(r) = \frac{n-1}{n+1}\,\frac{G\,m(r)}{r^2}. \tag{55}$$

These two equation leads to the ansatz of the solution:

$$Q(r) = c\left\{\,4\pi\, r^3\, \rho(r) - \frac{n-3}{n-1}\; m(r)\right\}. \tag{56}$$

Here, the variable c is an arbitrary constant. The above equation (56) automatically satisfies the boundary condition $Q(0) = 0$, while the remaining condition $Q(r_e) = 0$ puts the following constraint:

$$Q(r_e) = c\left(4\pi\, r_e^3\, \rho_e - \frac{n-3}{n-1}\, M\right) = c\left(\,u_e - \frac{n-3}{n-1}\,\right) M = 0. \tag{57}$$

Again, we arrive at the satisfactory result that the solution of zero-eigenvalue equation exactly recovers the condition (45).

Now, remaining task is to show that the second variation $\delta^2 F_q$ becomes negative beyond the transition point of $C_{\rm V} \to \pm\infty$. One can rewrite the expression (53) with

$$\delta^2 F_q = \frac{1}{2}\,(H-1)\int_0^{r_e} \frac{GQ^2}{r^2}\,dr,$$

with the constant H given by

$$H \;\equiv\; \frac{\displaystyle\frac{n+1}{n}\int_0^{r_e} \frac{1}{4\pi r^2 \rho}\left(\frac{P}{\rho}\right)\left(\frac{dQ}{dr}\right)^2 dr}{\displaystyle\int_0^{r_e} \frac{GQ^2}{r^2}\,dr}. \tag{58}$$

That is, the condition $H > 1$ implies stable local minimum state of free energy, while the inequality $H < 1$ represents unstable local maximum state. Integrating by part, equation (58) can be regarded as an eigenvalue equation with eigenvalue, H:

$$-\frac{d}{dr}\left\{\frac{1}{4\pi r^2\rho}\left(\frac{P}{\rho}\right)\frac{dQ}{dr}\right\} = H\ \frac{n}{n+1}\ \frac{GQ}{r^2}. \tag{59}$$

Obviously, equation (56) becomes the solution of above equation with the minimum eigenvalue, $H_{\rm min} = 1$, if the condition (57) is fulfilled. In this case, solution (56) can be regarded as the ground state of the eigensystem (59), since the function (56) does not possess any nodes between $[0, r_e]$. Therefore, for a suitably smaller radius r_e or a smaller density contrast ρ_e/ρ_c below the transition point, the eigenvalue H should be larger than unity. Conversely, from continuity, the condition $H < 1$ must be satisfied beyond the critical radius.

Finally, using the (u, v)-variables, the geometrical meaning of onset of thermodynamic instability is briefly discussed in similar manner to the adiabatic case. In Figure 4, the thick solid lines show the Emden trajectories with various polytrope indices in (u, v)-plane. The thin-solid lines in Figure 4 represents the straight lines, $u - (n-3)/(n-1) = 0$. Since the equilibrium state only exists along the Emden trajectory, the condition (57) is satisfied at the intersection of these two solid lines, which is only possible for $n > 3$. On the other hand, as seen in previous section, the equilibrium system surrounded by a thermal wall is characterized by the three parameters, r_e, M and β(or $T_{\rm phys}$), through the relation (31). In other words, the system must lie on the curve:

$$v = \left(\frac{\eta}{\alpha_n}\right)^{(n-1)/n} u^{-1/n}, \tag{60}$$

with some constant value η. We have seen in Figure 2 that the constant value η is bounded from above, $\eta \leq \eta_{\rm crit}$. Thus, the critical curve (60) with $\eta = \eta_{\rm crit}$ must intersect with both the Emden trajectory and the straight line $u - (n - 3)/(n-1) = 0$ simultaneously. This is clearly shown in Figure 4, where the critical curve is plotted as dashed lines. Since the critical curves tangentially intersect with Emden solutions, it always satisfies the condition $d\eta/d\xi = 0$ at the contact point, leading to the condition (45) consistently.

Table 1 summarizes the dimensionless quantities $\eta_{\rm crit}$ and $D_{\rm crit} \equiv (\rho_c/\rho_e)_{\rm crit}$ evaluated at the contact point. As increasing the polytrope index n, these values asymptotically approach the well-known results of Boltzmann-Gibbs limit, $\eta_{\rm crit} \to 2.52$ and $D_{\rm crit} \to 32.1$.

5 Origin of non-extensive nature in stellar polytrope

As has been mentioned in section 3.2, specific heat of the stellar polytropic system explicitly depends on the residual dimensional parameter h, in contrast to the isothermal limit (44). In this section, to contact the physical meaning of the non-extensivity in stellar polytrope, we discuss the origin of this residual dependence. Indeed, the appearance of the residual factor can be recognized as the breakdown of both the intensivity of temperature and the extensivity of energy and entropy as follows. From equation (18), the asymptotic behavior of the Emden solution becomes

$$\theta \sim \xi^{-2/(n-1)}, \qquad \rho \sim r^{-2n/(n-1)}, \qquad (\xi, r \to \infty)$$

so that the mass within a sphere of radius r is given by

$$M \sim \rho\, r^3 \propto r_e^{(n-3)/(n-1)} \quad . \tag{61}$$

Then the energy of a virialized stellar system is roughly estimated as

$$E \sim \frac{GM^2}{r_e} \propto r_e^{(n-5)/(n-1)} \propto M^{(n-5)/(n-3)},$$

and the relation (31) tells

$$\beta \propto r_e^{-(n-3)/(n-1)/(n-3/2)} \propto M^{-1/(n-3/2)}.$$

These relations clearly show the breakdown of the intensivity of temperature and the extensivity of energy, which lead to the scaling of the specific heat per mass:

$$\frac{C_{\rm V}}{N} = \frac{1}{M}\frac{dE}{dT_{\rm phys}} \sim \frac{\beta E}{M} \propto M^{-3(n-2)/(n-3)/(n-3/2)} \quad . \tag{62}$$

On the other hand, the dimensionless combination $h^2/(GMr_e)$ represents the ratio of a typical scale of the stellar system, $GMr \sim (GM/r)r^2 \sim v^2r^2$, to that of the reference cell, $h = v_0\, l_0$. This behaves as

$$\frac{h^2}{GMr_e} \propto \frac{1}{Mr_e} \propto M^{2(n-2)/(n-3)}. \tag{63}$$

Thus, these two equations (62) and (63) lead to the scaling relation of (42):

$$\frac{C_{\rm V}}{N} \sim \left(\frac{h^2}{GMr_e}\right)^{(3/2)/(n-3/2)}. \tag{64}$$

Notice that the Clausius relation (37) suggests that the entropy per unit mass has the same scaling relation:

$$\frac{S_q}{M} \sim \frac{\beta E}{M} \sim \frac{C_{\rm V}}{N},$$

Therefore, resultant dependence (64) for the stellar polytrope can be a natural outcome of the non-extensivity of the entropy.

In fact, framework of the thermostatistics generally requires an introduction of the scale of the unit cell in order to count the available number of states in phase spaces. This is even true in the case of the isothermal stellar system($n \to +\infty$ or $q \to 1$), but, the thermodynamic quantities show somewhat peculiar dependence of the scale h. A typical example is the entropy:

$$S_{\rm BG} = \frac{M}{m_0}\left\{\left(2u_e + v_e - \frac{9}{2}\right) - \ln\left(\frac{u_e v_e^{3/2}}{4\pi}\right) - \frac{3}{2}\ln\left(\frac{h^2}{2\pi GMr_e}\right)\right\},$$

where u_e and v_e are the homology invariants for the isothermal system. The above equation shows that in the Boltzmann-Gibbs limit, h-dependence of the entropy can be recognized as a matter of choice of an additive constant, so that its derivatives, e.g., specific heat, is free from the residual dependence.

It should be emphasized that the stellar equilibrium system recovers the extensivity in the limit $n \to \infty$ and it behaves as

$$E \sim M \sim r, \;\; C_{\rm V} \sim M. \tag{65}$$

Also, the temperature becomes intensive in this limit. Thus, we readily understand that the scaling behavior shown in (42) or (64) has nothing to do with the long-range nature of the gravity. Even in the free polytropic gas model in Appendix B, the residual dependence emerges as

$$\frac{C_{\rm V}}{N} \sim \left\{\frac{h^2}{(P/\rho)V^{2/3}}\right\}^{(3/2)/(n-3/2)}.$$

It follows that the explicit dependence of the specific heat on the reference cell scale h just originates from the the non-extensive nature of Tsallis entropy.

6 Summary & Discussions

In this paper, thermodynamic properties of the stellar self-gravitating system arising from Tsallis' non-extensive entropy have been studied in detail. In particular, physical interpretation of the thermodynamic instability previously found from the second variation of entropy is discussed in detail within a framework of the non-extensive thermostatistics. After briefly reviewing the equilibrium state of Tsallis entropy, we first address the issues on thermodynamic temperature in the case of equilibrium stellar polytrope. Analyzing the heat transfer and the entropy change in a quasi-static process, standard form of the Clausius relation is derived, irrespective of the non-extensivity of entropy. According to this result, we explicitly calculate the specific heat and confirm the presence of negative specific heat. The onset of instability found in previous work just corresponds to the zero-crossing point, $C_{\rm V} = 0$, supporting the fact that the heuristic explanation of gravothermal catastrophe holds even in the non-extensive thermostatistics.

Further, the analysis of specific heat shows divergent behavior at $n > 3$, suggesting another type of thermodynamic instability, which occurs when the system is surrounded by a thermal wall. We then turn to the stability analysis by means of the Helmholtz free energy. Similar to the previous early work, the stability/instability criterion just reduces to the solution of the zero-eigenvalue problem and solving the eigenvalue equation, we recover the marginal stability condition derived from the divergence of specific heat (45).

In addition to the thermostatistic treatment, we have also discussed the origin of non-extensivity in stellar polytrope. The residual dependence of the reference scale h appeared in the specific heat (42) naturally arises from the non-extensivity of the entropy and the resultant scaling dependence can be simply deduced from the asymptotic behavior of the Emden solutions.

The stability analysis using the free energy in section 4 is consistent with recent claim by Chavanis [23], who has investigated the dynamical instability of polytropic gas sphere. According to his early paper [22], the thermodynamic stability of stellar system is intimately related to the dynamical stability of gaseous system, which has been clearly shown in the case of the isothermal distribution. Thus, the correspondence between Chavanis' recent result [23] and a part of our present analysis can be regarded as a generalization of his early work to the polytropic system. Note, however, that starting from the Tsallis entropy, we extensively discuss the thermodynamic temperature and the specific heat of stellar polytrope. Therefore, at least, from the thermodynamic point of view, our present analysis provides a valuable insight to the stellar equilibrium systems.

A particular interest in the thermodynamic relation is the Clausius relation (37) that has been still preserved in the non-extensive stellar system. This is indeed consistent with the standard thermodynamic relation $\partial S/\partial E = \beta$, if one keeps the volume constant. Note also that the relation $\partial S/\partial E = \beta$ is readily obtained from the standard Legendre transform structure. While we only dealt with a specific case with the non-extensive entropy (5), it is well known that the standard Legendre transform structure does generally hold independently of the functional form of the entropy [17]. Hence, our result in turn suggests that the Clausius relation is also valid for any stellar system maximizing the entropic functional more general than Tsallis'.

At present, the results shown in this paper seems fully consistent with the general framework of the thermostatics. Apart from the thermodynamic instability, the stellar polytropic system can be a plausible thermodynamic equilibrium state, as well as the isothermal stellar distribution. In the isothermal case, existence of the thermodynamic limit has been discussed by de Vega and Sánchez [21]:

$$M,\ V \to \infty, \quad \frac{M}{V^{1/3}} = \text{fixed},$$

where $V \sim r^3$ is a volume of the system. Recalling the discussion in section 5, the above condition merely reflects the extensivity of the isothermal system (65). Thus, similar argument can hold for the non-extensive system. According to the scaling relation (61), the existence of the thermodynamic limit in stellar polytrope yields the condition:

$$M,\ V \to \infty, \quad \frac{M}{V^{(n-3)/(3n-3)}} = \text{fixed}.$$

Note, however, that this discussion relies on the non-uniqueness of the Boltzmann-Gibbs theory, which can be proven only mathematically[24]. Indeed, framework of the thermostatistics cannot answer the question whether the stellar polytropic distribution is really achieved as a thermodynamic equilibrium. To address this issue, we must study the detailed process of the long-term stellar dynamical evolution. In the light of this, the analysis using Fokker-Planck model or direct N-body simulation can provide an invaluable insight to the non-extensive nature of stellar gravitating systems. This issue is now in progress and will be presented elsewhere.

Another remaining issue is the re-examination of the present analysis from a view of the 'standard' Tsallis formalism using the normalized q-expectation values. Apart from some technical issues on the treatment of the maximum entropy principle, one might naively expect that the consistency between the

statistical and the thermodynamic analysis should be preserved even in the new formalism. However, a rather subtle point would be the identification of the thermodynamic temperature. As several author stated, the standard Clausius relation should be modified in the new Tsallis formalism and the resultant form of the expression apparently seems to contradict with the thermodynamic temperature defined through the thermodynamic zeroth law [15]. This point will be in particular important in discussing the thermodynamic instability and should be clarified along the line of our present treatment.

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Appendix A: Thermodynamic variables in a stellar polytropic system

In this appendix, using the equilibrium state of stellar polytrope described in section 2, we explicitly evaluate the thermodynamic variables, which have been used in section 3 and 4.

A.1 Energy

Recall that the equilibrium system confined in a spherical container satisfies the following virial theorem (e.g, p.502 of Ref.[3]):

$$2K + U = 4\pi r_e^3\, P_e.$$

The energy (1) is then expressed as

$$E = K + U = 4\pi r_e^3\, P_e - \; K = 4\pi r_e^3\, P_e - \frac{3}{2}\int_0^{r_e} P(r) 4\pi\, r^2 dr. \tag{66}$$

To evaluate the above integral in the spherically symmetric case, we use the following integral formula:

$$\int_0^{r_e} P(r)\, 4\pi r^2 dr = -\frac{1}{n-5}\left\{8\pi\, r_e^3\, P_e - (n+1)\,\frac{MP_e}{\rho_e} + \frac{GM^2}{r_e}\right\}, \tag{67}$$

which can be derived from the conditions of hydrostatic equilibrium, (15) and (16) (see Appendix A of ref.[2]). Thus, the energy of extremum state becomes

$$E = \frac{1}{n-5}\left[\frac{3}{2}\left\{\frac{GM^2}{r_e} - (n+1)\frac{MP_e}{\rho_e}\right\} + (n-2)\, 4\pi r_e^3 P_e\right]. \tag{68}$$

In terms of the homology invariants, we obtain

$$E = \frac{1}{n-5}\,\frac{GM^2}{r_e}\,\left[\frac{3}{2}\left\{1 - (n+1)\frac{1}{v_e}\right\} + (n-2)\frac{u_e}{v_e}\right]. \tag{69}$$

A.2 Entropy

First note the definition of Tsallis entropy (5):

$$S_q = -\left(n-\frac{3}{2}\right)\left\{\int N\left(\frac{f}{N}\right)^{(n-1/2)/(n-3/2)} d^6\boldsymbol{\tau} - N\right\}.$$

Substituting the distribution function (7) into the above equation, after some manipulation, we obtain

$$S_q = -\left(n-\frac{3}{2}\right)\left\{\beta\int P(x)\, d^3\boldsymbol{x} - N\right\}. \tag{70}$$

Thus, the substitution of integral formula (67) immediately leads to

$$S_q = \left(n-\frac{3}{2}\right)\left[\frac{1}{n-5}\left\{8\pi r_e^3 P_e - (n+1)\frac{MP_e}{\rho_e} + \frac{GM^2}{r_e}\right\}\beta + N\right],$$

which can be expressed in terms of the homology invariants:

$$S_q = \left(n-\frac{3}{2}\right)\left[\frac{1}{n-5}\,\frac{\beta GM^2}{r_e}\left\{2\frac{u_e}{v_e} - (n+1)\frac{1}{v_e} + 1\right\} + N\right]. \tag{71}$$

A.3 Radius-mass-temperature relation

The mass-radius-temperature relation (31) is derived from the equilibrium stellar polytropic configuration. Using (15), we first write down the condition of hydrostatic equilibrium at the boundary r_e:

$$\frac{GM}{r_e^2} = -\frac{1}{\rho_e}\left(\frac{dP}{dr}\right)_e.$$

The right-hand-side of this equation is rewritten with dimensionless quantities in (17):

$$\frac{GM}{r_e^2} = -(n+1)K_n\,\rho_c^{1/n}\left(\frac{\xi_e}{r_e}\right)\,\theta'_e. \tag{72}$$

We wish to express the above equation only in terms of the variables at the edge. To do this, we eliminate the residual dependences, ρ_c and K_n from (72). The definition (17) leads to

$$\frac{\xi_e}{r_e} = \left\{\frac{4\pi G\rho_c^2}{(n+1)P_c}\right\}^{1/2} = \left\{\frac{4\pi G}{(n+1)K_n}\right\}^{1/2}\rho_c^{(n-1)/(2n)},$$

which can be rewritten with

$$\rho_c^{1/n} = \left\{ \frac{4\pi G}{(n+1)K_n} \right\}^{1/(n-1)} \left(\frac{\xi_e}{r_e} \right)^{2/(n-1)} .$$

Substituting the above relation into (72), the ρ_c-dependence is first eliminated and we obtain

$$\frac{G^{n/(n-1)} M}{r_e^{(n-3)/(n-1)}} = - \left[\frac{\{(n+1)K_n\}^n}{4\pi} \right]^{1/(n-1)} \xi_e^{(n+1)/(n-1)} \, \theta'_e. \tag{73}$$

As for K_n-dependence, the definition (13) together with (8) yields

$$(n+1)\, K_n = \left\{ 4\sqrt{2}\pi \, \frac{B(3/2, n-1/2)}{(n-1)^{n-3/2}} \, \frac{M}{h^3} \right\}^{-1/n} (m_0\beta)^{-(n-3/2)/n}. \tag{74}$$

Hence, substituting the above expression into (73), the relation between mass M, radius r_e and Lagrange multiplier β can be finally obtained. In terms of the homology invariants, it follows that

$$\left\{ \frac{(GM)^n (m_0\beta)^{n-3/2}}{r_e^{n-3} h^3} \right\}^{1/(n-1)} = \alpha_n \, (u_e \; v_e^n)^{1/(n-1)} , \tag{75}$$

where the constant α_n is given by

$$\alpha_n \equiv \left\{ \frac{(n-1/2)^{n-3/2}}{16\sqrt{2}\pi^2 \, (n+1)^n \, B(3/2, n-1/2)} \right\}^{1/(n-1)} ,$$

which asymptotically approaches unity in the limit $n \to \infty$.

Appendix B: Thermodynamic temperature of classical gas model from the Carnot cycle

In a standard framework of thermodynamics, the temperature is defined by means of an efficiency of the Carnot cycle. Here we apply the standard procedure to seek the physical temperature $T_{\rm phys}$ for so-called polytropic system of which distribution function is given by the extremization of the Tsallis entropy (see eqs.(5)(6)). For simplicity, we discuss a case of the free classical gas without gravity, which corresponds to the $G \to 0$ limit of the *stellar polytropic system*.

From the $G \to 0$ limit of the formula (68), free polytropic system of the volume V with *homogeneous* pressure P and density ρ has an (internal) energy:

$$E = K = \frac{3}{2} PV = \frac{3}{2} \frac{MP}{\rho}. \tag{76}$$

Here we drop the subscript $_e$ for the pressure and density, since both are constant within the system in absence of gravity. And equation of state (11) becomes

$$P = K_n \, \rho^{1+1/n} \; = K_n \left(\frac{M}{V} \right)^{1+1/n}. \tag{77}$$

From equations (8) and (13), the constant K_n is related to the Lagrange multiplier β as

$$K_n \propto \beta^{-(n-3/2)/n}, \tag{78}$$

so that this constant can be used as a parameter which characterizes the temperature of the system. However, it is not sure whether K_n itself has a role of the physical temperature, which should be determined through the efficiency of the Carnot cycle.

The internal energy (76) and the equation of state (77) give the thermodynamic first law:

$$\begin{aligned} d'Q &= dE + \, P \, dV \\ &= M^{1+1/n} \left\{ \frac{3}{2} \frac{dK_n}{V^{1/n}} + \left(\frac{n-3/2}{n} \right) K_n \, \frac{dV}{V^{1+1/n}} \right\}, \end{aligned} \tag{79}$$

from which adiabatic changes $d'Q = 0$ is expressed as

$$K_n \, V^{(2/3-1/n)} = \text{constant}, \qquad P \, V^{5/3} = \text{constant}'. \tag{80}$$

Note that adiabatic lines in a P-V plane become steeper than isothermal ones when $n > 3/2$.

Now, let us consider the Carnot cycle shown in Figure 5. As usual, quasi-static changes B $\to$ C and D $\to$ A are adiabatic. As for the process A $\to$ B, the system is in a thermal contact with a heat bath which has a higher temperature K_n^H. Similarly, during the change C $\to$ D, the system lies in a thermal equilibrium with another heat bath that has a lower temperature K_n^L. The system absorbs amount of heat Q^H from the higher temperature bath and

disposes Q^L to the lower one during the isothermal processes A→ B and C→ D, respectively. They are easily evaluated from (79):

$$
\begin{aligned}
Q^H &= (n - \tfrac{3}{2})\, M^{1+1/n}\; K_n^H\, \left(V_A^{-1/n} - V_B^{-1/n}\right), \\
Q^L &= (n - \tfrac{3}{2})\, M^{1+1/n}\; K_n^L\, \left(V_D^{-1/n} - V_C^{-1/n}\right).
\end{aligned} \tag{81}
$$

On the other hand, a relation between the parameters of the cycle can be obtained from the equation of state (77) and the adiabatic changes (80):

$$
\left(\frac{K_n^H}{K_n^L}\right)^{\gamma} = \frac{V_C}{V_B} = \frac{V_D}{V_A}\ ; \quad \gamma = \frac{3}{2}\,\frac{n}{n-3/2}. \tag{82}
$$

Thus, equations (81) and (82) lead to the following efficiency of the Carnot cycle:

$$
\eta \equiv 1 - \frac{Q^L}{Q^H} = \ 1\ -\ \left(\frac{K_n^L}{K_n^H}\right)^{n/(n-3/2)} = \ 1\ -\ \frac{\beta^H}{\beta^L}, \tag{83}
$$

where we used the relation (78) in the last line. This clearly shows that the inverse of the Lagrange multiplier β has a role of the physical temperature.

References

[1] C. Tsallis, J.Stat.Phys. 52 (1988) 479.

[2] A. Taruya, M. Sakagami, Physica A 307 (2002) 185.

[3] J. Binney, S. Tremaine, *Galactic Dynamics* (Princeton Univ. Press, Princeton, 1987).

[4] R. Elson, P. Hut and S. Inagaki, Ann. Rev. Astron. Astrophys. 25 (1987) 565.

[5] G. Meylan, D.C. Heggie, Astron.Astrophys.Rev. 8 (1997) 1.

[6] V.A. Antonov, *Vest. Leningrad Gros. Univ.*, 7 (1962) 135 (English transl. in *IAU Symposium 113, Dynamics of Globular Clusters*, ed. J. Goodman and P. Hut [Dordrecht: Reidel], pp. 525–540 [1985])

[7] D. Lynden-Bell, R. Wood, Mon.Not.R.Astr.Soc. 138 (1968) 495.

[8] T. Padmanabhan, Astrophys.J.Suppl. 71 (1989) 651.

[9] T. Padmanabhan, Phys.Rep. 188 (1990) 285.

[10] C. Tsallis, Braz. J. Phys. 29 (1999) 1.

[11] S. Abe, Y. Okamoto (Eds.), *Nonextensive Statistical Mechanics and Its Applications* (Springer, Berlin, 2001)

[12] A.R. Plastino, A. Plastino, Phys.Lett. A 174 (1993) 384.

[13] A.R. Plastino, A. Plastino, Braz. J. Phys. 29 (1999) 79.

[14] S. Martínez, F. Nicolás, F. Pennini, A. Plastino, Physica A 286 (2000) 489.

[15] S. Abe, S. Martínez, F. Pennini, A. Plastino, Phys.Lett. A 281 (2001) 126.

[16] E.M.F. Curado, C. Tsallis, J.Phys.A 24 (1991) L69.

[17] A. Plastino, A.R. Plastino, Phys.Lett. A 226 (1997) 257.

[18] C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534.

[19] S. Chandrasekhar, *Introduction to the Study of Stellar Structure* (New York, Dover, 1939)

[20] R. Kippenhahn, A. Weigert, *Stellar Structure and Evolution* (Springer, Berlin, 1990)

[21] H.J. de Vega, N. Sánchez, Nucl.Phys. B 625 (2002) 409.

[22] P.H. Chavanis, Astron. & Astrophys. 381 (2002) 340.

[23] P.H. Chavanis, astro-ph/0108378.

[24] S. Abe, A.K. Rajagopal, Phys.Lett. A 272 (2000) 345; J. Phys. A 33 (2000) 8733; Europhys. Lett. 52 (2000) 610.

Table 1
Critical values of the radius-mass-temperature relation, $\eta_{\rm crit}$ and the density contrast between center and edge, $D_{\rm crit} = (\rho_c/\rho_e)_{\rm crit}$ in the case of a system in contact with a heat bath for given polytrope index n or q.

n	q	$\eta_{\rm crit}$	$D_{\rm crit}$
3	$\frac{5}{3}$	—	—
4	$\frac{7}{5}$	0.9421	153.5
5	$\frac{9}{7}$	1.193	88.15
6	1.22	1.379	68.38
7	1.18	1.520	58.86
8	1.15	1.631	53.28
9	1.13	1.720	49.62
10	1.12	1.793	47.04
30	1.04	2.263	35.89
50	1.02	2.363	34.28
100	1.01	2.440	33.17
∞	1	2.518	32.13

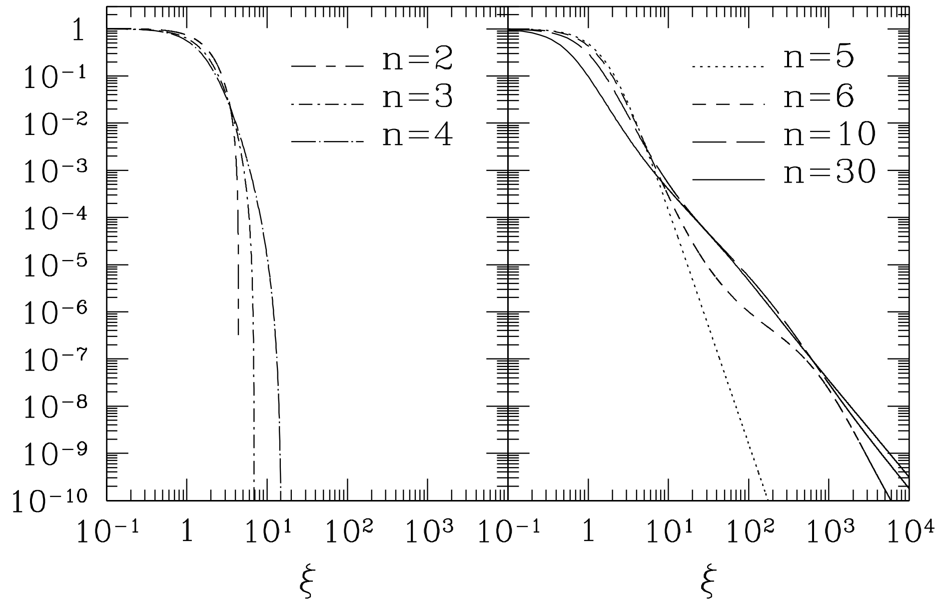

Fig. 1. Density profiles of stellar polytrope for $n < 5$ (*left*) and $n \geq 5$(*right*).

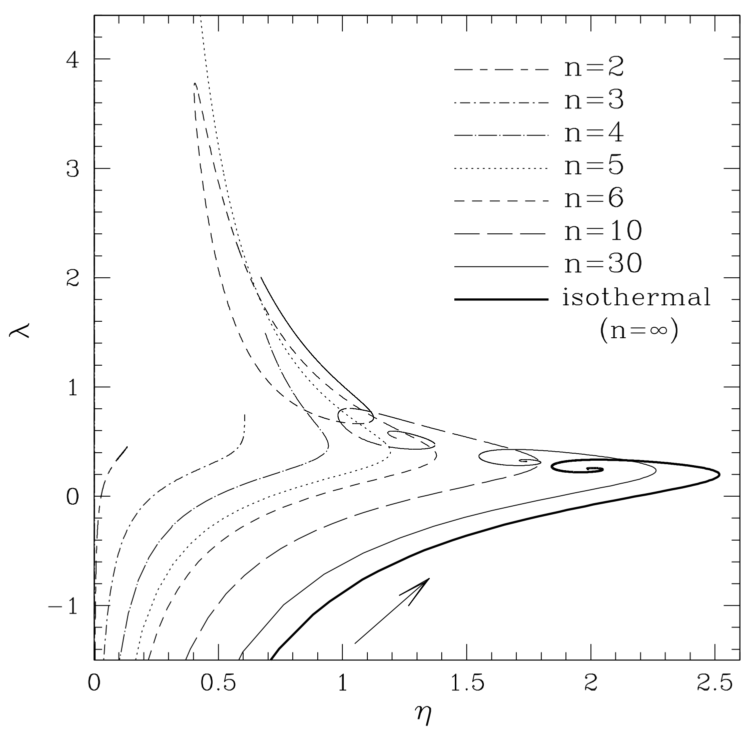

Fig. 2. Trajectory of Emden solutions in (η, λ)-plane.

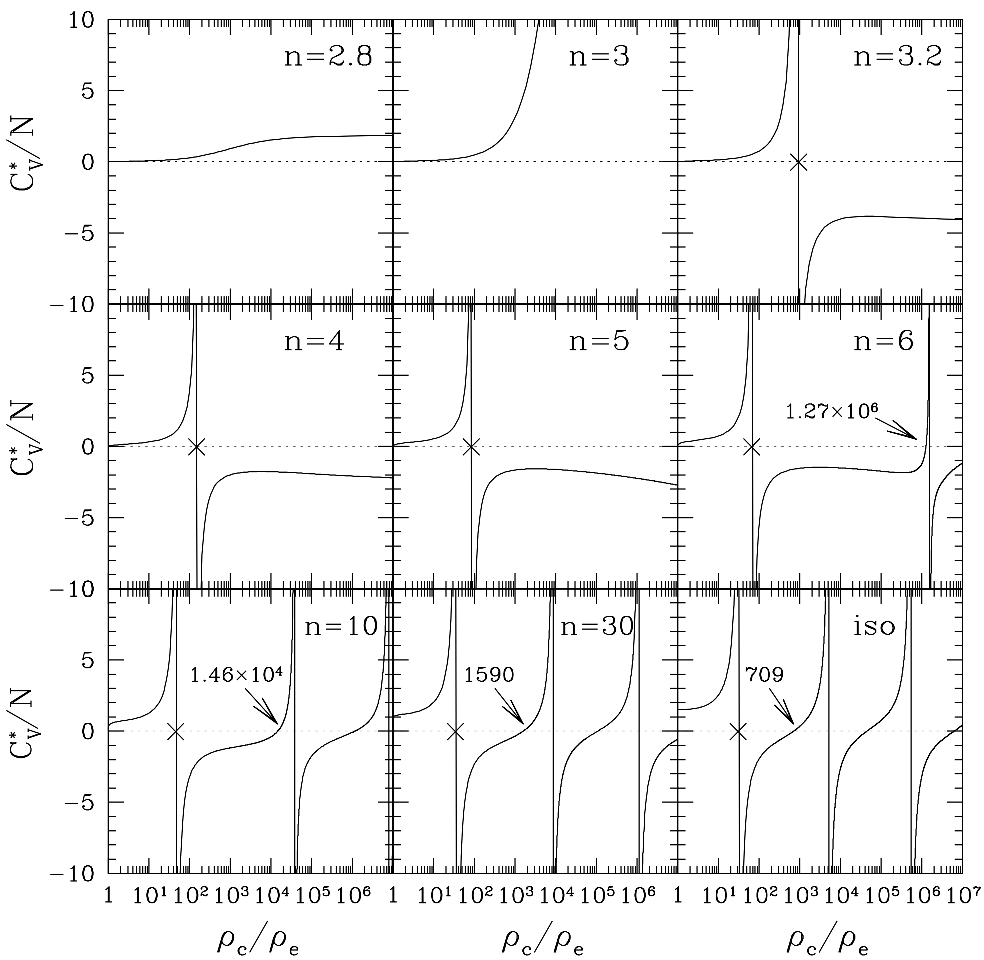

Fig. 3. Normalized specific heat per particle C_V^*/N as a function of density contrast ρ_c/ρ_e near the critical polytrope indices $n = 3$(*upper*) and $n = 5$(*middle*), and large n cases(*lower*). Here, the normalized specific heat C_V^* is defined by $C_V/(h^2/GMr_e)^{(3/2)/(n-3/2)}$.

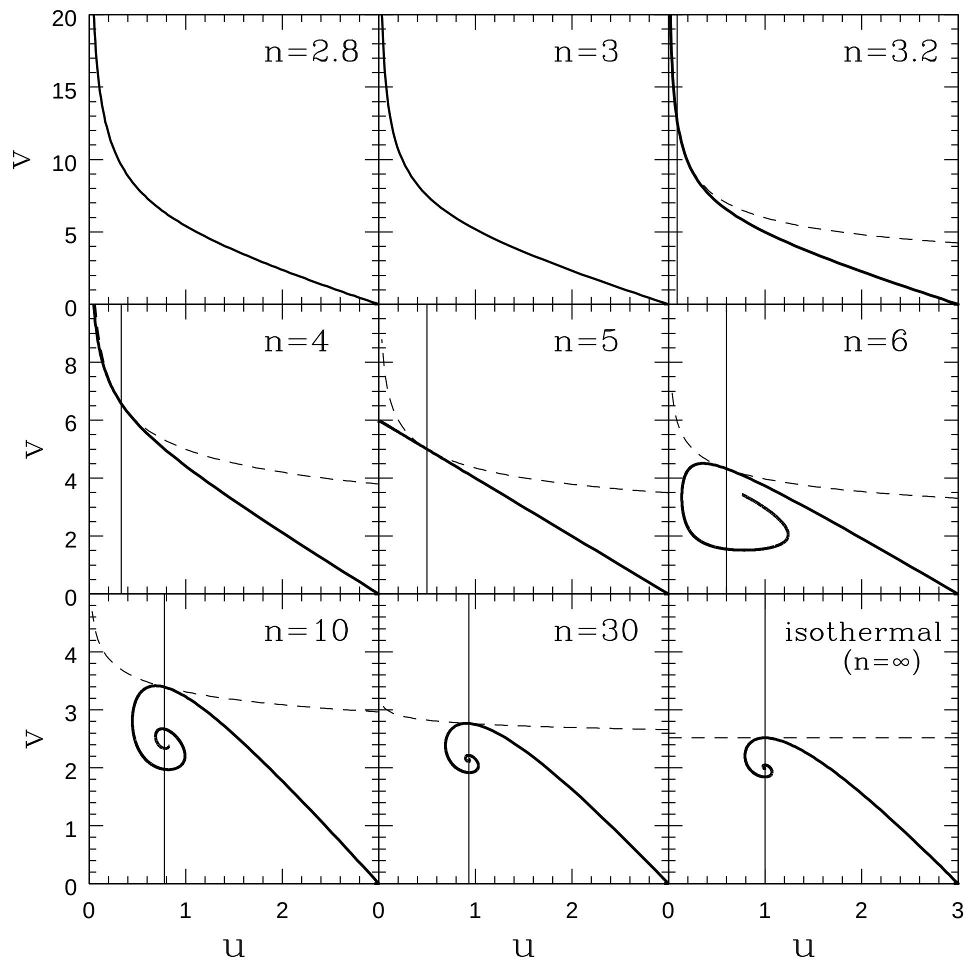

Fig. 4. Stability/instability criterion for a system in contact with a thermal bath in (u, v)-plane. The thick solid lines represent the trajectories of Emden solutions, while the thin-solid and dashed lines respectively denote the conditions (45) and (60).

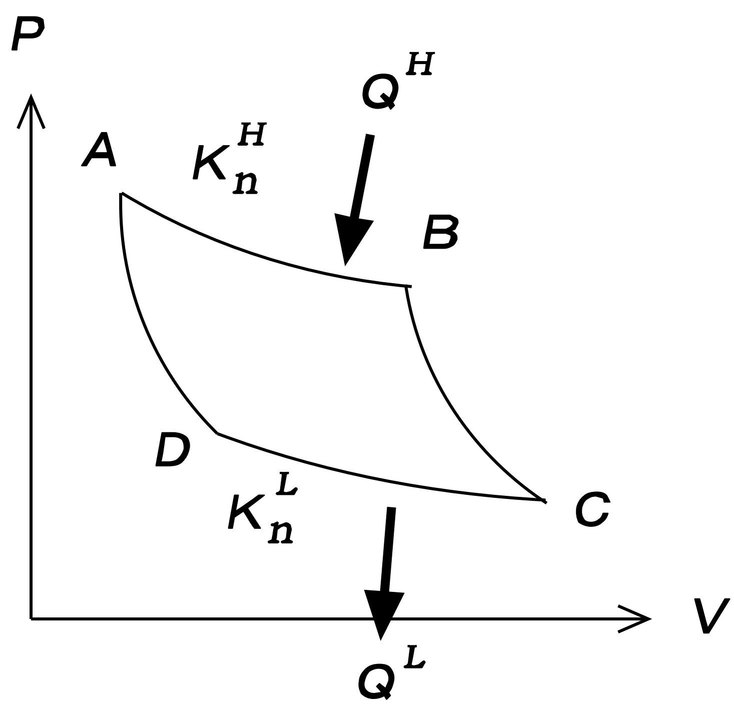

Fig. 5. A schematic description of Carnot cycle.